[13] K. H. Ra, "Etude et rèlisation de dispositif à couplage inter grilles à effet de champ," Master's thesis, INP Toulouse, July 1981.
[14] J. B. Davies, "A least-square boundary residual method for the numerical solution of scattering problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-21, pp. 99–104, Feb. 1973.
[15] H. J. A. Lariviere and J. B. Davies, "The solution of electromagnetic eigen value problems by least squares boundary residuals," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 436–441, May 1975.
[16] F. B. Llewellyn, "Some fundamental properties of transmission systems," *Proc. IRE*, vol. 40, pp. 271–283, Mar. 1952.
[17] E. Folke Bolinder, "Survey of some properties of linear networks," *IRE Trans. Circuit Theory*, vol. CT-4, pp. 70–78, Sept. 1957.
[18] J. M. Rollet, "Stability and power gain in variants of linear two ports," *IRE Trans. Circuit Theory*, vol. CT-9, pp. 29–32, Mar. 1962.

Henri Baudrand was born in 1939. He obtained the "Diplôme d'Ingénieur" in electronics and the Docteur-es-Sciences degree in microwaves, both from E.N.S.E.E.I.H.T., Toulouse, France, in 1962 and 1966, respectively.

Since then, he has been working on active and passive microwave integrated circuits in the "Laboratoire de Microondes" of E.N.S.E.E.I.H.T. He is a Professor of Physics in this school.

Tanos El Khoury photograph and biography not available at the time of publication.

Désiré Lilonga was born in Ekouli, Congo, in 1956. He obtained the "Maîtrise de Physique" from the Pau University in 1980 and the "Diplôme d'Etudes Approfondies" in physics from the Bordeaux University in 1981.

He then joined Laboratoire de Microonde in Toulouse, France, where he has been working on the coupling between the space-charge wave in semiconductors and an interdigital structure.

The Electrostatic Field of Conducting Bodies in Multiple Dielectric Media

SADASIVA M. RAO, MEMBER, IEEE, TAPAN K. SARKAR, SENIOR MEMBER, IEEE, AND ROGER F. HARRINGTON, FELLOW, IEEE

Abstract —**A method for computing the electrostatic fields and the capacitance matrix for a multiconductor system in a multiple dielectric region is presented. The number of conductors and the number of dielectrics in this analysis are arbitrary. Some of the conductors may be of finite volume and others may be infinitesimally thin. The conductors can be either above a single ground plane or between two parallel ground planes. The formulation is obtained by using a free-space Green's function in conjunction with total charge on the conductor-to-dielectric interfaces and polarization charge on the dielectric-to-dielectric interfaces. The solution is effected by the method of moments using triangular subdomains with piecewise constant expansion functions and point matching for testing. Computed results are given for some finite-length conducting lines, compared to previous results obtained by two-dimensional analysis.**

Manuscript received March 5, 1985; revised May 29, 1984. This work has been supported by the Digital Equipment Corp., Marlboro, MA 01752.

S. M. Rao and T. K. Sarkar are with the Department of Electrical Engineering, Rochester Institute of Technology, Rochester, NY 14623.

R. F. Harrington is with the Department of Electrical Engineering, Syracuse University, Syracuse, NY 13210.

I. Introduction

THE OBJECTIVE of this paper is to compute the electrostatic fields and the capacitance matrix of arbitrarily shaped conductors embedded in multiple dielectric regions. The entire system could be situated over a finite or infinite ground plane, or could be between two ground planes. This solution is useful for finding equivalent circuits of microstrip junctions and discontinuities and for vias connecting conductors located in various dielectric regions. Some of the conductors may be of finite volume and others may be infinitesimally thin.

Recent advances in integrated circuit technology, such as VLSI design in the microwave region, necessitate a sophisticated analysis, design, and construction of transmission lines to carry signals from one end to the other. Even though a large volume of literature exists to analyze an infinitely long transmission line, there are very few satisfactory procedures to solve for the equivalent circuits

0018-9480/84/1100-1441$01.00 ©1984 IEEE

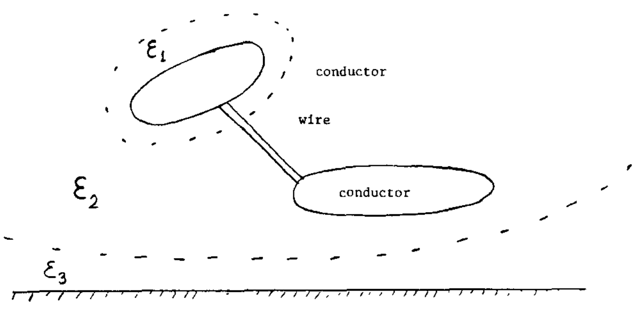


Fig. 1. Multiconductor system embedded in a multilayered dielectric region above a ground plane.

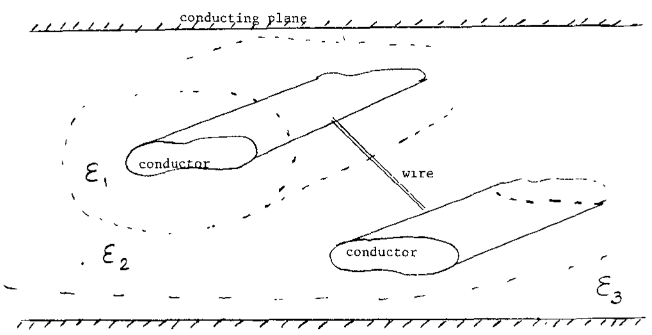


Fig. 2. Multiconductor transmission system embedded in a multilayered dielectric region between two ground planes.

of junctions and vias connecting conductors located in various dielectric layers. The basic difference between the analysis of transmission lines and junctions of lines, vias, etc., is that for the former case we are solving a two-dimensional problem and, for the latter case, we are solving a three-dimensional problem. The primary difficulty in solving three-dimensional problems arises from the lack of an effective solution procedure to treat arbitrarily shaped conducting bodies (like microstrip discontinuities and vias) immersed in several dielectric media. The simpler case of treating a single, arbitrarily shaped, conducting body immersed in a homogeneous dielectric medium has been solved by Rao *et al.* [1] and in [2]. In this paper, we present an extension of the solution procedure to treat multiple finite conductors in the presence of various dielectric media.

The present paper uses a free-space Green's function in conjunction with total charge on the conductor-to-dielectric interfaces and polarization charge on the dielectric-to-dielectric interfaces. This approach is similar to the approach presented in [3] for the two dimensional case. The free-space Green's function approach results in a simpler formulation of the problem, but requires the solution of a larger matrix equation. This formulation has a theoretical advantage over those presented in [4]–[8] in that there is no limit to the number of dielectric layers that can be treated, but a practical limit is imposed by the speed and the storage of the computer. Also, using this approach, one can handle any number of arbitrarily shaped conductors.

II. Statement of the Problem

Consider a system of conductors in a multiple dielectric region above a ground plane as shown in either Fig. 1 or Fig. 2. The system is of finite volume. An arbitrary number N_c of perfect conductors are embedded in an arbitrary number N_d of dielectric layers. Some of the conductors may be of finite cross section. Others may be infinitesimally thin strips that appear as curves. The permittivity of the jth dielectric is ϵ_j. In Fig. 1, the uppermost dielectric extends to infinity. In Fig. 2, there is an upper ground plane.

A lower ground plane is present in both Figs. 1 and 2. This lower ground plane is taken to be infinite in extent. Nominally, the upper ground plane and the dielectric layers are also of infinite extent. However, the numerical solution of Section IV is obtained by considering the upper ground plane and the dielectric layers to be of finite extent.

The objective is to determine the capacitance matrix of the multiple conductors embedded in a multiple dielectric region. The ijth element of the capacitance matrix is the free charge on the ith conductor when the potential of the jth conductor is 1 V and the other conductors are grounded. In [3], the elements of the capacitance matrix are called coefficients of capacitance. In [9, p. 97], the diagonal elements of the capacitance matrix are called coefficients of capacitance, but the off-diagonal elements are called coefficients of induction.

Once the capacitance and inductance matrices of the multiple conductor system are known, the equivalent circuit can be determined. However, in this paper, examples are presented only for the transmission-line systems. In our examples, we consider the transmission lines to be of finite length and sufficiently long so that we can compare our results with those of the two-dimensional results.

III. Analysis

Consider the capacitance matrix for the problem stated in the previous section. The ijth element of this matrix is the free charge on the ith conductor when all conductors except the jth conductor are grounded and the jth conductor is charged to a potential of 1 V. Hence, the elements of the capacitance matrix can be determined by relating the free charge on the conductors to the potentials of the conductors. The free charge on one of the N_c conductors is the integral of the free charge per unit area over the entire surface of the conductor. Thus, the elements of the capacitance matrix can easily be determined once a relationship has been established between the total free charge on the surfaces of the conductors and the potentials of the conductors.

A total charge of σ_T per unit area is assumed on the conductor-to-dielectric interfaces and the $N_d - 1$ dielectric-to-dielectric interfaces. The conductor-to-dielectric interfaces consist of the surfaces of the N_c conductors and the upper ground plane, if present. The jth dielectric-to-dielectric interface is where the dielectric layers ϵ_j and ϵ_{j+1} meet, provided that no conductors lie on this plane. If conductors lie on this surface, then the jth dielectric-to-dielectric interface is the portion of this surface not occupied by conductors. On each conductor-to-dielectric interface, the total charge is the sum of free charge and polarization charge. On each dielectric-to-dielectric interface, the total charge is polarization charge.

We discretize each charge density layer into a set of planar triangular patches and approximate the charge density to be constant on a given patch. The triangular patches to model a layer are chosen primarily because of the capability of these patches to conform to any geometrical surface and boundary. Moreover, these patches permit greater patch densities at the regions where more resolution is desired and the planar triangular scheme is easily described to the computer.

At any point $\boldsymbol{r}$ above the lower ground plane, the potential V is due to the combination of σ_T on all the patches and the image of σ_T about the lower ground plane. Hence

$$V(\boldsymbol{r})=\frac{1}{4\pi\epsilon_0}\sum_{j=1}^{J}\int_{S_j}\sigma_T(\boldsymbol{r}')\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-\frac{1}{|\boldsymbol{r}-\hat{\boldsymbol{r}}'|}\right]dS' \quad (1)$$

where S_j is the surface area of the jth patch. In (1), dS' is the differential element of area at $\boldsymbol{r}'$ on S_j and $\hat{\boldsymbol{r}}'$ is the image of $\boldsymbol{r}'$ about the lower ground plane. The first N_c interfaces are the surfaces of the N_c conductors. If there is no upper ground plane, the next N_d-1 interfaces are the dielectric-to-dielectric interfaces. If there is an upper ground plane, the (N_c+1)th interface is the surface of this ground plane, and the next N_d-1 interfaces are the dielectric-to-dielectric interfaces. Accordingly

$$J=J_1+J_2 \quad (2)$$

where, in the absence of the upper ground plane

$$J_1=N_c$$
$$J_2=N_d-1 \quad (3)$$

and, in the presence of the upper ground plane

$$J_1=N_c+1$$
$$J_2=N_d-1. \quad (4)$$

It is evident that J_1 is the number of conductor-to-dielectric interfaces and that J_2 is the number of dielectric-to-dielectric interfaces.

The electric field $\boldsymbol{E}$ is given by

$$\boldsymbol{E}(\boldsymbol{r})=-\nabla V(\boldsymbol{r}). \quad (5)$$

Substituting (1) for V in (5) and assuming that $\boldsymbol{r}$ is not on any of the patches $\{S_j\}$ so that the ∇ operator may be taken under the integral sign, we obtain

$$\boldsymbol{E}(\boldsymbol{r})=\frac{1}{4\pi\epsilon_0}\sum_{j=1}^{J}\int_{S_j}\sigma_T(\boldsymbol{r}')\left[\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}-\frac{\boldsymbol{r}-\hat{\boldsymbol{r}}'}{|\boldsymbol{r}-\hat{\boldsymbol{r}}'|^3}\right]dS'. \quad (6)$$

Taking the limit of (6) as $\boldsymbol{r}$ approaches the interface S_i, we obtain the following formula for $\boldsymbol{E}$, valid on S_i.

$$\boldsymbol{E}^{\pm}(\boldsymbol{r})=\frac{1}{4\pi\epsilon_0}\sum_{j=1}^{J}\int_{S_j}\sigma_T(\boldsymbol{r}')\left[\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}-\frac{\boldsymbol{r}-\hat{\boldsymbol{r}}'}{|\boldsymbol{r}-\hat{\boldsymbol{r}}'|^3}\right]dS' \pm\boldsymbol{n}\frac{\sigma_T(\boldsymbol{r})}{2\epsilon_0},\quad \begin{cases}\boldsymbol{r}\text{ on }S_i\\ i=1,2,\cdots,J.\end{cases} \quad (7)$$

Here, $\boldsymbol{n}$ is the unit vector normal to S_i at $\boldsymbol{r}$. The side of S_i toward which $\boldsymbol{n}$ points is called the positive side of S_i. The side of S_i away from which $\boldsymbol{n}$ points is called the negative side of S_i. In (7), $\boldsymbol{E}^+(\boldsymbol{r})$ is the electric field on the positive side of S_i and $\boldsymbol{E}^-(\boldsymbol{r})$ is the electric field on the negative side of S_i.

On each conductor-to-dielectric interface, the potential is constant. Denoting the potential on the ith conductor-to-dielectric interface by V_i, we obtain

$$V(\boldsymbol{r})=V_i,\quad \begin{cases}\boldsymbol{r}\text{ on }S_i\\ i=1,2,\cdots,J_1\end{cases}. \quad (8)$$

If the upper ground plane is present, then V_i is zero for $i=J_1$. Substitution of (1) for $V(\boldsymbol{r})$ in (8) yields

$$\frac{1}{4\pi\epsilon_0}\sum_{j=1}^{J}\int_{S_j}\sigma_T(\boldsymbol{r}')\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-\frac{1}{|\boldsymbol{r}-\hat{\boldsymbol{r}}'|}\right]dS'=V_i,\quad \begin{cases}\boldsymbol{r}\text{ on }S_i\\ i=1,2,\cdots,J_1\end{cases}. \quad (9)$$

The displacement vector is called $\boldsymbol{D}(\boldsymbol{r})$. The normal component of $\boldsymbol{D}(\boldsymbol{r})$ is continuous across each dielectric-to-dielectric interface. Since $\boldsymbol{D}(\boldsymbol{r})$ is the product of permittivity with electric field, it follows that

$$\epsilon_{i-J_1}\boldsymbol{E}^+(\boldsymbol{r})\cdot\boldsymbol{n}=\epsilon_{i+1-J_1}\boldsymbol{E}^-(\boldsymbol{r})\cdot\boldsymbol{n},\quad \begin{cases}\boldsymbol{r}\text{ on }S_i\\ i=J_1+1,J_1+2,\cdots,J\end{cases} \quad (10)$$

where $\boldsymbol{n}$ is the outward-pointing unit vector. In (10), ϵ_{i-J_1} and $\boldsymbol{E}^+(\boldsymbol{r})$ are, respectively, the permittivity and electric field on the side of S_i into which $\boldsymbol{n}$ points. Moreover, ϵ_{i+1-J_1} and $\boldsymbol{E}^-(\boldsymbol{r})$ are, respectively, the permittivity and electric field on the other side of S_i. Substitution of (7) for $\boldsymbol{E}^{\pm}(\boldsymbol{r})$ in (10) yields, after division by $(\epsilon_{i-J_1}-\epsilon_{i+1-J_1})$

$$\frac{(\epsilon_{i-J_1}+\epsilon_{i+1-J_1})}{2\epsilon_0(\epsilon_{i-J_1}-\epsilon_{i+1-J_1})}\sigma_T(\boldsymbol{r})+\frac{1}{4\pi\epsilon_0}\sum_{j=1}^{J}\int_{S_j}\sigma_T(\boldsymbol{r}')\left[\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}-\frac{\boldsymbol{r}-\hat{\boldsymbol{r}}'}{|\boldsymbol{r}-\hat{\boldsymbol{r}}'|^3}\right]\cdot\hat{\boldsymbol{n}}\,dS'=0,\quad \begin{cases}\boldsymbol{r}\text{ on }S_i\\ i=J_1+1,J_1+2,\cdots,J\end{cases}. \quad (11)$$

Equations (9) and (11) are a set of J integral equations in the unknown total charge σ_T on the interfaces whose patches are $\{S_j, j=1,2,\cdots,J\}$. In Section IV, the method of moments will be used to obtain an approximate numerical solution for σ_T in terms of $\{V_i, i=1,2,\cdots,N_c\}$. Since (9) and (11) are linear, this solution is of the form

$$\sigma_T = \sum_{i=1}^{N_c} \sigma_T^{(i)} V_i \tag{12}$$

where $\sigma_T^{(i)}$ is the solution which would result if the potential V_i was unity and all other potentials were zero.

As stated earlier, some of the conductors may be of finite cross section, and others may be infinitesimally thin. On the ith triangular patch on a conductor

$$\sigma_T = \epsilon_0 \boldsymbol{E}\cdot\boldsymbol{n} \tag{13}$$

$$\sigma_F = \epsilon \boldsymbol{E}\cdot\boldsymbol{n} \tag{14}$$

where $\boldsymbol{E}$ is the electric field just outside the patch, $\boldsymbol{n}$ is the unit normal vector which points outward from the surface of the conductor, ϵ is the permittivity just outside the conductor, and σ_F is the free charge per unit area on the conductor. Equations (13) and (14) imply that

$$\sigma_F(\boldsymbol{r}) = \frac{\epsilon(\boldsymbol{r})}{\epsilon_0}\sigma_T(\boldsymbol{r}) \tag{15}$$

on the surface of the ith conductor, provided that this conductor is of finite cross section.

If the ith conductor is infinitesimally thin, then the free charge σ_F on the surface of the ith conductor is given by

$$\sigma_F = (\epsilon^+ \boldsymbol{E}^+ - \epsilon^- \boldsymbol{E}^-)\cdot\boldsymbol{n} \tag{16}$$

where $\boldsymbol{n}$ is a unit vector normal to the conductor. The side of the conductor toward which $\boldsymbol{n}$ points is called the positive side. The side of the conductor away from which $\boldsymbol{n}$ points is called the negative side. In (16), ϵ^+ and $\boldsymbol{E}^+$ are, respectively, the permittivity and electric field on the positive side of the conductor. Moreover, ϵ^- and $\boldsymbol{E}^-$ are, respectively, the permittivity and electric field on the negative side of the conductor. Substitution of (7) for $\boldsymbol{E}^{\pm}$ in (16) leads to

$$\sigma_F(\boldsymbol{r}) = \frac{\epsilon^+(\boldsymbol{r})+\epsilon^-(\boldsymbol{r})}{2\epsilon_0}\sigma_T(\boldsymbol{r}) + \frac{\epsilon^+(\boldsymbol{r})-\epsilon^-(\boldsymbol{r})}{4\pi\epsilon_0} \cdot \sum_{j=1}^{J}\int_{S_j}\sigma_T(\boldsymbol{r}')\left[\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} - \frac{\boldsymbol{r}-\hat{\boldsymbol{r}}'}{|\boldsymbol{r}-\hat{\boldsymbol{r}}'|^3}\right]\cdot\boldsymbol{n}\,dS \tag{17}$$

on the surface of the ith conductor provided that this conductor is an infinitesimally thin conductor.

Regardless of whether the ith conductor has finite cross section or is infinitesimally thin, the free charge Q_i on it is given by

$$Q_i = \int_{S_i}\sigma_F(\boldsymbol{r})\,dS, \qquad i=1,2,\cdots,N_c \tag{18}$$

where dS is the differential area at $\boldsymbol{r}$ on patch S_i. In view of (12) with index i replaced by j, substitution of (15) or (17) for σ_F in (18) gives

$$Q_i = \sum_{j=1}^{N_c} C_{ij}V_j, \qquad i=1,2,\cdots,N_c \tag{19}$$

where, if the ith conductor is of finite cross section

$$C_{ij} = \int_{S_i}\frac{\epsilon(\boldsymbol{r})}{\epsilon_0}\sigma_T^{(j)}(\boldsymbol{r})\,dS. \tag{20}$$

If the ith conductor is infinitesimally thin, then

$$C_{ij} = \int_{S_i}\left\{\frac{\epsilon^+(\boldsymbol{r})+\epsilon^-(\boldsymbol{r})}{2\epsilon_0}\sigma_T^{(j)}(\boldsymbol{r}) + \frac{\epsilon^+(\boldsymbol{r})-\epsilon^-(\boldsymbol{r})}{4\pi\epsilon_0} \cdot \sum_{k=1}^{J}\int_{S_k}\sigma_T^{(j)}(\boldsymbol{r}')\left[\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} - \frac{\boldsymbol{r}-\hat{\boldsymbol{r}}'}{|\boldsymbol{r}-\hat{\boldsymbol{r}}'|^3}\right]\cdot\boldsymbol{n}\,dS'\right\}dS. \tag{21}$$

In obtaining (21), the index j in (17) was replaced by k in order to avoid confusion with the index j which appears in C_{ij}. The coefficient C_{ij} is the ijth element of the capacitance matrix.

IV. Development of the Moment Solution

In this section, the integral equations (9) and (11) are solved numerically for σ_T by means of the method of moments [11].

A solution σ_T to (9) and (11) is sought in the form

$$\sigma_T(\boldsymbol{r}) = \sum_{n=1}^{N}\sigma_{Tn}P_n(\boldsymbol{r}) \tag{22}$$

where $\{P_n(\boldsymbol{r}), n=1,2,\cdots,N\}$ are pulse expansion functions given by

$$P_n(\boldsymbol{r}) = \begin{cases} 1, & \boldsymbol{r} \text{ in the triangular patch } S_n \\ 0, & \text{otherwise} \end{cases}. \tag{23}$$

Moreover, $\{\sigma_{Tn}, n=1,2,\cdots,N\}$ are constants to be determined. The upper ground plane and dielectric layers are now truncated at a finite distance so that only pulse functions of finite domain are needed. Given an arbitrary point on the truncated $\{S_j, j=1,2,\cdots,J\}$, there is an integer m such that, at this point

$$P_m = 1$$
$$P_n = 0, \qquad n=1,2,\cdots,m-1,m+1,\cdots,N. \tag{24}$$

It follows from (23) and (24) that

$$\sigma_T = \sigma_{Tm} \tag{25}$$

at this point.

Let $\{P_n(\boldsymbol{r}), n=1,2,\cdots,N_1\}$ be the pulses on $\{S_j, j=1,2,\cdots,J_1\}$, and let $\{P_n(\boldsymbol{r}), n=N_1+1,N_1+2,\cdots,N\}$ be the pulses on $\{S_j, j=J_1+1,J_1+2,\cdots,J\}$. Moreover, let $\boldsymbol{r}_m$ be the centroid of the domain of $P_m(\boldsymbol{r})$ for $m=1,2,\cdots,N$.

Substituting (22) for σ_T in (9) and then enforcing (9) at $\boldsymbol{r}=\boldsymbol{r}_m$ for $m=1,2,\cdots,N_1$, we obtain

$$\sum_{n=1}^{N} Z_{mn}\sigma_{Tn} = V_i, \qquad m=1,2,\cdots,N_1 \tag{26}$$

where i is such that $\boldsymbol{r}_m$ is on S_i, and

$$Z_{mn}=\frac{1}{4\pi\epsilon_0}\int_{\Delta S_n}\left[\frac{1}{|\boldsymbol{r}_m-\boldsymbol{r}'|}-\frac{1}{|\boldsymbol{r}_m-\hat{\boldsymbol{r}}'|}\right]dS'\cdot\begin{cases}m=1,2,\cdots,N_1\\ n=1,2,\cdots,N\end{cases}\tag{27}$$

where ΔS_n is the triangular domain of $P_n(\boldsymbol{r})$.

Substituting (22) for σ_T in (11) and then enforcing (11) at $\boldsymbol{r}=\boldsymbol{r}_m$ for $m=N_1+1,\ N_1+2,\cdots,N$, we obtain

$$\sum_{n=1}^{N} Z_{mn}\sigma_{Tn}=0,\qquad m=N_1+1,N_1+2,\cdots,N\tag{28}$$

where $m\neq n$, and

$$Z_{mn}=\frac{1}{4\pi\epsilon_0}\int_{\Delta S_n}\left[\frac{\boldsymbol{r}_m-\boldsymbol{r}'}{|\boldsymbol{r}_m-\boldsymbol{r}'|^3}-\frac{\boldsymbol{r}_m-\hat{\boldsymbol{r}}'}{|\boldsymbol{r}_m-\hat{\boldsymbol{r}}'|^3}\right]\cdot\boldsymbol{n}\,dS',\quad\begin{cases}m=N_1+1,N_1+2,\cdots,N\\ n=1,2,\cdots,N.\end{cases}\tag{29}$$

In (28), Z_{mm} is given by

$$Z_{mm}=\frac{\epsilon_{i-J_1}+\epsilon_{i+1-J_1}}{2\epsilon_0(\epsilon_{i-J_1}-\epsilon_{i+1-J_1})}+\frac{1}{4\pi\epsilon_0}\int_{\Delta S_m}\left(\frac{\boldsymbol{r}_m-\boldsymbol{r}'}{|\boldsymbol{r}_m-\boldsymbol{r}'|^3}\right)\cdot\boldsymbol{n}\,dS'-\frac{1}{4\pi\epsilon_0}\int_{\Delta S_m}\left(\frac{\boldsymbol{r}_m-\hat{\boldsymbol{r}}'}{|\boldsymbol{r}_m-\hat{\boldsymbol{r}}'|^3}\right)\cdot\boldsymbol{n}\,dS',$$
$$m=N_1+1,N_1+2,\cdots,N.\tag{30}$$

In (30), i is such that $\boldsymbol{r}_m$ is on S_i. If $m\neq n$, but if $\boldsymbol{r}_m$ and P_n are on the same dielectric-to-dielectric interface, then (29) reduces to

$$Z_{mn}=-\frac{1}{4\pi\epsilon_0}\int_{\Delta S_n}\left(\frac{\boldsymbol{r}_m-\hat{\boldsymbol{r}}'}{|\boldsymbol{r}_m-\hat{\boldsymbol{r}}'|^3}\right)\cdot\boldsymbol{n}\,dS'.\tag{31}$$

Numerical methods for calculating Z_{mn} are given later in this section. After Z_{mn} has been calculated for $m=1,2\cdots N$ and $n=1,2,\cdots,N$, (26) and (28) combine to form N simultaneous equations in the N unknowns $\{\sigma_{Tn}, n=1,2,\cdots,N\}$. These simultaneous equations can then be solved for $\{\sigma_{Tn}, n=1,2,\cdots,N\}$ in terms of $\{V_i, i=1,2,\cdots,N_c\}$. The solution is of the form

$$\sigma_{Tn}=\sum_{i=1}^{N_c}\sigma_{Tn}^{(i)}V_i\tag{32}$$

where $\{\sigma_{Tn}^{(i)}, n=1,2,\cdots,N\}$ is the solution which would result if V_i were unity and all other V's were zero. Substituting (32) and (22) and comparing the result with (12), we obtain

$$\sigma_T^{(i)}(\boldsymbol{r})=\sum_{n=1}^{N}\sigma^{(i)}P_n(\boldsymbol{r}).\tag{33}$$

The elements of the capacitance matrix can be calculated by replacing i by j in (33) and then substituting the resulting expression for $\sigma_T^{(j)}(\boldsymbol{r})$ in (20) and (21). The integral with respect to S in (21) is approximated by sampling the integrand at $\boldsymbol{r}=\boldsymbol{r}_m$ for all values of m for which $\boldsymbol{r}_m$ is on S_j. At $\boldsymbol{r}=\boldsymbol{r}_m$, the integrals with respect to S' in (21) are similar to the integrals appearing in expressions (29) and (30) for Z_{mn}.

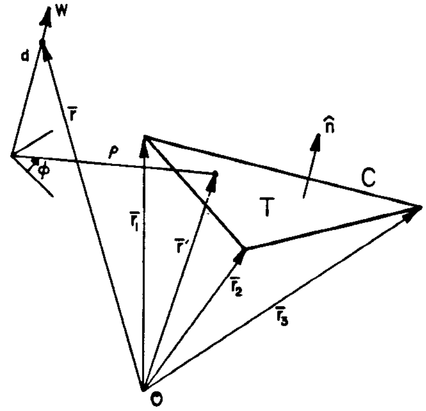


Fig. 3. Geometry associated with observation point and triangular source region.

To facilitate calculation of Z_{mn}, ΔS_n is approximated by triangular patches as shown in Fig. 3. Each of the integrals in (27) has been evaluated analytically over the triangular-shaped regions. The details are given in [13], [14].

The integrations for the terms in (29), however, have been done numerically utilizing the techniques described in [12].

V. Numerical Examples

A computer program has been written for the special case where all conductors are of finite cross section and for the case where all conductors are infinitesimally thin. In all the numerical computations, the conductors and the dielectric-to-dielectric interfaces have a finite area. These programs [13], [14] have been used to obtain the numerical results presented in this section.

In our examples, we treat only finite-length transmission lines and compare our results with the two-dimensional case. We do this not because of limitations introduced by the computer programs but because results for two-dimensional examples are readily available. In order to find the inductance matrix for our computations of finite-length multiconductor transmission line, we recompute the capacitance matrix $[C_0]$ with no dielectric media and then use the relationship

$$[L]=\frac{1}{v^2}[C_0]^{-1}\tag{34}$$

where v is the velocity of light in free space. The above relation is exact for the two-dimensional case [3], whereas it provides a simple approximate procedure for the three-dimensional case (finite-length conductors).

Example 1

Consider the case of a commonly used microstrip transmission line with two rectangular conducting cylinders over a dielectric slab on an infinitely long perfect ground plane as shown in Fig. 4. The relative permittivity ϵ_r of the dielectric slab is 2.0. The two-dimensional solution for this problem was reported by Weeks [17]. For the three-dimen-

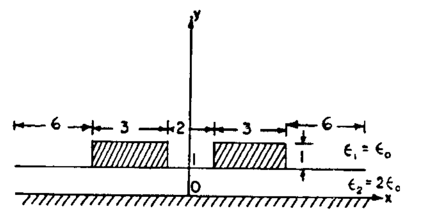


Fig. 4. Coupled transmission line formed by two conductors on dielectric substrate over a ground plane.

TABLE I
CAPACITANCE AND INDUCTANCE MATRICES FOR THE TWO-CONDUCTOR TRANSMISSION LINE

	Our Result	Weeks [17]
C(1,1) = C(2,2)	9.189 x 10^{-11} F/m	9.224 x 10^{-11} F/m
C(1,2) = C(2,1)	-7.019 x 10^{-12} F/m	-8.504 x 10^{-12} F/m
L(1,1) = L(2,2)	0.1855 x 10^{-6} H/m	0.1982 x 10^{-6} H/m
L(1,2) = L(2,1)	0.2361 x 10^{-7}	0.2980 x 10^{-6} H/m

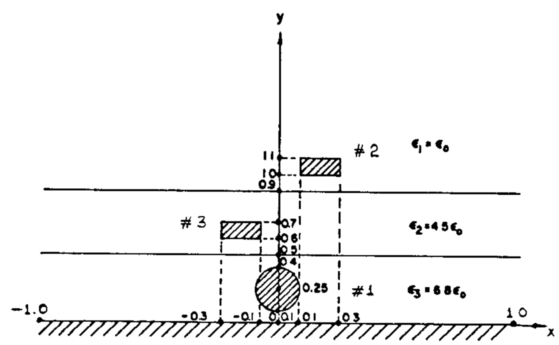


Fig. 5. Three-conductor transmission line over a ground plane.

TABLE II
CAPACITANCE MATRIX FOR THREE-CONDUCTOR TRANSMISSION LINE

Capacitance F/m	Our Result	Two Dimensional Analysis [3]
C(1,1)	0.3984 x 10^{-9}	0.3523 x 10^{-9}
C(1,2)	-0.6659 x 10^{-10}	-0.6825 x 10^{-10}
C(1,3)	-0.6730 x 10^{-11}	-0.7196 x 10^{-11}
C(2,1)	-0.6970 x 10^{-10}	-0.6825 x 10^{-10}
C(2,2)	0.1399 x 10^{-9}	0.1244 x 10^{-9}
C(2,3)	-0.1221 x 10^{-10}	-0.1300 x 10^{-10}
C(3,1)	-0.7649 x 10^{-11}	-0.7196 x 10^{-11}
C(3,2)	-0.1290 x 10^{-10}	-0.1300 x 10^{-10}
C(3,3)	0.3815 x 10^{-10}	0.3340 x 10^{-10}

sional problem, we take each conductor to be 20.0 m long and each is modeled by utilizing 40 triangular patches. The two dielectric layers at the end of the striplines are each 20 m long and 6.0 m wide and are modeled by 16 patches each. The dielectric section between the two conductors is 20.0×2.0 m and is modeled by eight triangular patches. Table I shows the values of C_{ij} and L_{ij} obtained from our solution compared with those taken from [17]. Satisfactory agreement is evident in these two solutions.

Example 2

Consider the case of a microstrip transmission line as shown in Fig. 5. Here, we have a circular conductor and two rectangular conductors embedded in three dielectric regions of relative permittivity 6.8, 4.5, and 1.0, respectively. The circular conductor is 2 m long and is modeled by using 80 triangular patches. The rectangular conductors are each 2 m long and are modeled utilizing 40 triangular patches. Each dielectric layer is 2×2 m and is modeled by 32 triangular patches. The entire structure is situated over an infinitely long perfect ground plane. This example illustrates the generality of the solution procedure to handle various geometries. Table II shows the capacitance matrix obtained from our solution compared with the two-dimensional analysis presented in [3].

Example 3

Consider the case of two rectangular microstrip lines situated between two perfectly conducting ground planes. The lower ground plane is considered infinitely long, whereas the upper ground plane is considered to be of finite length of dimension 20×20 m. This geometry is shown in Fig. 6. Each rectangular conductor is modeled by 40 triangular patches and is 20 m long. The upper ground plane is considered to be a dielectric of $\epsilon_r = 99$, so as to approximate the case of a ground plane. This layer has been modeled utilizing 72 triangular patches. Table III shows a comparison of the results computed with the present program and that of Weeks [17] and Wei *et al.* [3]. Note that the results of Weeks are for conductors between two perfectly conducting infinitely long ground planes, whereas those of Wei *et al.* [3] are for a finite dielectric layer ($\epsilon_r = 99$) on top. Our results are for a three-dimensional case, whereas the other two are for a two-dimensional case. Again, the computed results appear reasonable.

Example 4

Consider the case of a commonly used microstrip transmission line with two infinitely thin strips over a dielectric slab on an infinitely long perfectly conducting ground plane as shown in Fig. 7. The relative permittivity ϵ_r of the dielectric slab is 9.6. The two-dimensional solution for this

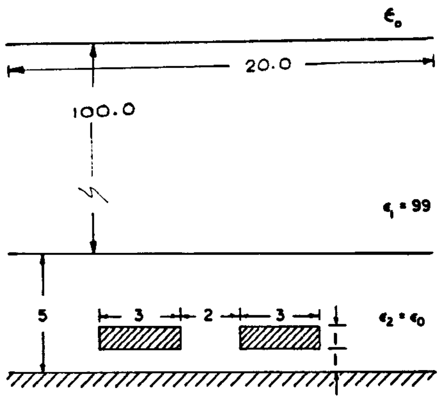


Fig. 6. Coupled microstrips between parallel conducting planes.

TABLE III
COMPARISON OF RESULTS FOR FIG. 6

	Weeks [17]	Wei et al [3]	Our Program
C_{11}	0.6307×10^{-10}	0.6233×10^{-10}	0.6509×10^{-10}
C_{12}	-0.5866×10^{-11}	-0.5931×10^{-11}	-0.5777×10^{-11}
C_{21}	-0.5866×10^{-11}	-0.5931×10^{-11}	-0.5783×10^{-11}
C_{22}	0.6307×10^{-10}	0.6233×10^{-10}	0.6516×10^{-10}

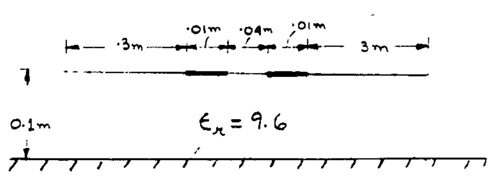


Fig. 7. A parallel strip transmission line over a dielectric slab mounted on a ground plane.

problem was reported by Bryant and Weiss [19]. For the three-dimensional solution of this problem, we take each strip to be 1.0 m long and model by utilizing 16 triangular patches. The two dielectric layers at the end of the strip-lines are each 10 m long and 0.3 m wide and are modeled by 16 triangular patches each. The dielectric section between the two conductors is 0.04 m wide and 1.0 m long, and also modeled by 16 triangular patches. The even-mode and odd-mode impedances Z_{oe} and Z_{oo} for our case are 140.25 and 80.40 Ω, respectively, whereas the two-dimensional solution yields 141.2 and 77.25 Ω, respectively [19]. Again, satisfactory agreement is evident in these two solutions.

VI. DISCUSSION AND CONCLUSION

The integral equations (9) and (11) for the total charge at the surfaces of conductors embedded in multiple dielectric regions and on the dielectric-to-dielectric interfaces are simple in concept. The solution obtained by the method of moments using a piecewise constant for expansion and point matching for testing is also simple. Experience has shown that this type of solution is both versatile and accurate. Improvement in the rate of convergence might be obtained by using better-behaved functions for expansion and testing, but at the cost of considerable complication.

The solution presented is valid for an arbitrary number of conductors of finite volume embedded in an arbitrary number of dielectrics. The solution is also valid, even if the conductors become of zero thickness.

For the numerical examples, we have compared the inductance and capacitance matrices of a finite-length transmission line with that of an infinitely long transmission line. However, this analysis is more general and can be used to compute equivalent circuits of microstrip junctions and discontinuities and for vias connecting signal conductors.

Even though the method is designed for general geometries, its application seems rather restricted for complex configurations, due to solutions of large matrix equations. However, if one applied the conjugate gradient method [20], [21] directly on the operator equations (9) and (11), then one needs only $5N$ storage locations as opposed to N^2 for conventional matrix methods.

REFERENCES

[1] S. M. Rao, A. W. Glisson, D. R. Wilton, and B. S. Vidula, "A simple numerical solution procedure for statics problems involving arbitrary shaped surfaces," *IEEE Trans. Antennas Propagat.*, vol. AP-27, no. 5, pp. 604–608, Sept. 1979.

[2] J. J. H. Wang and Charles J. Drane, "Numerical analysis of arbitrarily shaped bodies modeled by surface patches," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-30, pp. 1167–1173, Aug. 1982.

[3] C. Wei, R. F. Harrington, J. R. Mautz, and T. K. Sarkar, "Multiconductor transmission lines in multilayered dielectric media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 439–450, Apr. 1984.

[4] P. Benedek and P. Silvester, "Capacitance of parallel rectangular plates separated by a dielectric sheet," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 504–510, Aug. 1972.

[5] A. Farrar and A. T. Adams, "Computation of lumped microstrip capacities by matrix methods—Rectangular sections," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 495–497, May 1971.

[6] D. Sinnott, "Application of the numerical solution to Laplaces's equation in three dimensions," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 135–136, Feb. 1968.

[7] Y. L. Chow, "An approximate dynamic Green's function in three dimensions for finite length microstrip line," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 393–397, Apr. 1980.

[8] W. C. Chew and J. A. Kong, "Effects of fringing fields on the capacitance of circular microstrip disk," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 98–104, Feb. 1980.

[9] R. Plonsey and R. E. Collin, *Principles and Applications of Electromagnetic Fields.* New York: McGraw Hill, 1961.

[10] D. Kajkez, "Multiconductor transmission lines," Interaction note 151, AFWL (EL), Kirtland AFB, NM, Carl Baum, Ed., June 1972.

[11] R. F. Harrington, *Field Computation by Moment Methods.* New York: Macmillan, 1968; reprinted by Krieger, Melbourne, FL, 1982.

[12] P. C. Hammer, O. P. Marlowe, and A. H. Stroud, "Numerical integration over simplexes and cores," *Math. Tables Aids Comp.*, vol. 10, pp. 130–137, 1956.

[13] S. M. Rao and T. K. Sarkar, "Analysis of arbitrarily oriented three dimensional microstrip lines," Dept. Elec. Eng., Rochester Inst. Technol., Rochester, NY, Jan. 1983.

[14] S. M. Rao and T. K. Sarkar, "Analysis of infinitely thin, finite length microstrip transmission lines," Dept. Elec. Eng., Rochester Inst. Technology, Rochester, NY, Mar. 1983.

[15] M. Javid and P. M. Brown, *Field Analysis and Electromagnetics.* New York: McGraw-Hill, 1963.

[16] S. Saad, R. C. Hansen, and G. J. Wheeler, *Microwave Engineers Handbook*, Vol. I. Dedham, MA: Artech House, 1971.

[17] W. T. Weeks, "Calculation of coefficients of capacitance of multiconductor transmission lines in the presence of dielectric interface," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 35–43, Jan. 1970.

[18] C. E. Smith and R. S. Chang, "Microstrip transmission line with finite-width dielectric," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, Feb. 1980.

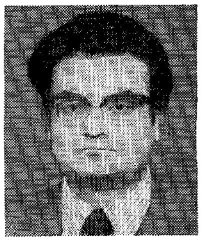
[19] T. G. Bryant and J. A. Weiss, "MSTRIP (Parameters of microstrip)," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, Apr. 1971.

[20] T. K. Sarkar and S. M. Rao, "The application of the conjugate gradient method for the solution of electromagnetic scattering from wire antennas," *IEEE Trans. Antennas Propagat.*, to be published.

[21] T. K. Sarkar, "The application of the conjugate gradient method for the solution of operator equations arising in electromagnetic scattering from wire antennas," *Radio Sci.*, Sept. 1984, to be published.

Sadasiva M. Rao (M'83) received the Bachelors degree in electrical and communication engineering from Osmania University, Hyderabad, India, in 1974, and the Masters degree in 1976 from the Indian Institute of Sciences, Bangalore, India, and the Ph.D. degree from the University of Mississippi, University, in 1980.

Since 1976, he has been a Research Assistant in the Department of Electrical Engineering at the University of Mississippi, University. Currently, he is an Assistant Professor at Rochester Institute of Technology, Rochester, NY. His research interests are in the areas of electromagnetic theory and numerical methods applied to antennas and scattering.



Tapan K. Sarkar (S'69–M'76–SM'81) was born in Calcutta, India, on August 2, 1948. He received the B. Tech. degree from the Indian Institute of Technology, Kharagpur, India, in 1969, the M.Sc.E. degree from the University of New Brunswick, Fredericton, Canada, in 1971, and the M.S. and Ph.D. degrees from Syracuse University, Syracuse, NY, in 1975.

From 1969 to 1971, he served as an instructor at the University of New Brunswick. While studying at Syracuse University, he served as an Instructor and Research Assistant in the Department of Electrical and Computer Engineering, where is is presently an Adjunct Assistant Professor. Since 1976, he has been an Assistant Professor at the Rochester Institute of Technology, Rochester, NY. From 1977 to 1978, he was a Research Fellow at the Gordon McKay Laboratory of Harvard University, Cambridge, MA. His current research interests deal with system identification, signal processing, and analysis of electrically large electromagnetic systems.

Dr. Sarkar is a member of Sigma Xi and URSI Commission B.

Roger F. Harrington (S'48–A'53–M'57–SM'62–F'68) was born in Buffalo, NY, on December 24, 1925. He received the B.E.E. and M.E.E. degrees from Syracuse University, Syracuse, NY, in 1948 and 1950, respectively, and the Ph.D. degree from Ohio State University, Columbus, OH, in 1952.

From 1945 to 1946, he served as an Instructor at the U.S. Naval Radio Materiel School, Dearborn, MI, and from 1948 to 1950, he was employed as an Instructor and Research Assistant at Syracuse University. While studying at Ohio State University, he served as a Research Fellow in the Antenna Laboratory. Since 1952, he has been on the faculty of Syracuse University, where he is presently Professor of Electrical Engineering. During 1959–1960, he was Visiting Associate Professor at the University of Illinois, Urbana, in 1964 he was Visiting Professor at the University of California, Berkeley, and in 1969 he was Guest Professor at the Technical University of Denmark, Lyngby, Denmark.

Dr. Harrington is a member of Tau Beta Pi, Sigma Xi, and the American Association of University Professors.